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Electromagnetic momentum, magnetic model of light and effects of the Aharonov-Bohm type

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Abstract. The wave equation for light propagation in slowly moving media, which is analogous to that of quantum effects of the Aharonov-Bohm type, is characterized by the interaction momentum **Q**, related to the flow **u**. In effects of the Aharonov-Bohm type the interaction momentum **Q** is related to the momentum of the electromagnetic (em) fields, that characterizes an em flow **u**. It is shown that in both cases **Q** has the same physical origin. Calculation of the interaction em momentum **Q** for the light wave dragged by the flow yields exactly the Fresnel-Fizeau momentum. These results corroborate the validity of the magnetic model for light and highlight the role and relevance of the em momentum in new effects of classical and quantum physics. A tentative test of an astrophysical Fizeau-Aharonov-Bohm effect is discussed.

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1 Introduction

Wave propagation in moving media has attracted the attention of several physicists in recent years. The analogy between the wave equation for light in moving media and that for charged matter waves has been pointed out by Hannay [1] and later addressed by Cook, Fearn, and Milonni [2] who have suggested that light propagation at a fluid vortex is analogous to the Aharonov-Bohm (AB) effect, where charged matter waves (electrons) encircle a localized magnetic flux [3]. Generally, in quantum effects of the AB type [3–8] matter waves undergo an electromagnetic (em) interaction as if they were propagating in a flow of em origin that acts as a moving medium [6] and modifies the wave velocity. However, quantum effects are not restricted to em interaction: e.g., in the one considered by Colella, Overhauser, and Werner [9], the interaction is not em but gravitational.

Several interesting connections between the optics of moving media and other fields of physics involving wave propagation have been pointed out. A magnetic model of light propagation in moving media and its relativistic theory, has been elaborated by Leonhardt and Piwnicki [10]. We recall that, according to Fresnel [11], light waves propagating in a transparent, incompressible moving medium with uniform refraction index n, are dragged by the medium and develop an interference structure that depends on the velocity $\mathbf{u}$ of the fluid. The speed achieved is

$$v = \frac{c}{n} + \left(1 - \frac{1}{n^2}\right) u \tag{1}$$

as later corroborated by Fizeau [12].

In effects of the AB type [3–8] and in the propagation of light in moving media there is em interaction between waves and medium. However, the magnetic model of waves is not restricted to light propagation and has been extended to water waves by Berry et al. [13] who report both the theory and an experiment related to it. Acoustical analogues have been described and observed in moving classical media [14] and should be visible in superfluids [15]. Besides the connection between the Fizeau effect and the AB effect in a real material medium, it has been shown that a non-uniformly moving medium appears to light as an effective gravitational field for which the curvature scalar is nonzero [10,16]. Other applications that involve the magnetic model of light describe slow-light pulses in moving media [17] while an analogue of the Fizeau effect for massive and massless particles in an effective optical medium has been derived from the static, spherically symmetric gravitational field [18].

The magnetic model of light is based on the existing formal analogy between the non-relativistic expression of the wave equation for light in moving media and the Schrödinger equation for charged matter waves in the presence of the external vector potential $\mathbf{A}$, i.e. the magnetic AB effect. In both cases the waves interact with the external medium so that the wave equations described in

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Section 2 contain a term that is generically referred to as the interaction momentum $\mathbf{Q}$. The wave function Ψ for light is a component of the classical em field, while for matter waves Ψ stands for the usual quantum wave function. Furthermore, the interaction momentum for the light wave is related to the flow $\mathbf{u}$, while the AB interaction term $e\mathbf{A}/c$ is proportional to the vector potential $\mathbf{A}$, and a priori the physical link between the two terms is not clear. Thus, the existing analogy between the two wave equations can either be a formal similitude, devoid of a deeper physical meaning, or else a physically meaningful analogy that involves an interaction of the same physical nature.

One of the purposes of this article is to show that the em interactions involved do indeed possess the same physical origin that is a common feature in the two equations. The discussion is set within a wider context that includes not only the traditional AB effect but all effects of the AB type. Furthermore, we show in Sections 3 and 4 that, in analogy with matter waves of effects of the AB type, the interaction momentum $\mathbf{Q}$ for light in moving media is related to the linear momentum of the em fields $\mathbf{P}_e$. The momentum $\mathbf{Q}$ is linked to the variation of the em momentum $\mathbf{P}_e$ of the light ray due to the flow $\mathbf{u}$. Finally, in Section 5 the momentum $\mathbf{Q}$ is related to the polarization of the medium and is calculated as the net variation of the interaction polarization em momentum due the flow $\mathbf{u}$, i.e., the dragged interaction em momentum, a task that some physicist has considered unattainable. The value of $\mathbf{Q}$ calculated for light waves yields exactly the Fresnel-Fizeau momentum foreseen by special relativity. This result corroborates and generalizes the magnetic model of light, validates and clarifies its physical origin and provides further support to the several applications mentioned above which are based on it. In closing, the possibility of testing a Fizeau-Aharonov-Bohm effect for photons at an astrophysical scale is discussed in a qualitative and tentative way in Section 6.

As discussed and elaborated in the conclusions, the em momentum provides a common link for the classical and quantum effects [5–8] mentioned above and assumes a predominant, unitary role in determining the *interaction* Hamiltonians in classical and quantum physics. In fact, we allege that these new physical results have been achieved thanks to the approach based on the em momentum.

2 Wave equations for matter and light waves

The magnetic model for light in moving media is based on the formal analogy of the wave equation for light with the wave equation for matter (Schrödinger equation) [1,2,10]. However, although matter and light waves share the same wave equation they do not share the same Hamiltonian. In fact, as shown below, the Hamiltonian for material particles (usually, electrons) with rest mass m_o is the Hamiltonian H_{AB} of effects of the AB type, while the Hamiltonian for light rays (photons) is $H = \hbar\omega$ where for light rays or photons we have $m_o = 0$. Although electrons and photons do not necessarily exhibit the same behavior, the fact that the corresponding waves share the same wave equation implies a close analogy for the behavior of matter and light waves, as discussed in the following sections.

2.1 Matter waves

In quantum effects of the AB type [3–8], a beam of interfering particles possessing em properties interacts with external em fields and potentials in a force-free (or field-free) region of space. These effects are nonlocal in the sense that there are no external forces acting locally on the particles so that an important characteristic is that, despite the em interaction, the particle momentum $\mathbf{p} = m\mathbf{v}$ and energy $E = (1/2)mv^2$ is conserved.

In seeking an analogy between the equations for matter waves and light waves, we conveniently write the Schrödinger equation for quantum effects of the AB type as

$$\left(-i\nabla - \frac{\mathbf{Q}}{\hbar}\right)^2 \Psi = \frac{p^2}{\hbar^2}\Psi, \tag{2}$$

where $p^2 = 2mE$. Its solution is given by the matter wave function

$$\Psi = e^{i\phi}\Psi_0 = e^{i\frac{1}{\hbar}\int \mathbf{Q}\cdot d\mathbf{x}}\Psi_0 = e^{i\frac{1}{\hbar}\int \mathbf{Q}\cdot d\mathbf{x}}\, e^{i\frac{1}{\hbar}(\mathbf{p}\cdot\mathbf{x}-Et)}\,\mathcal{A} \tag{3}$$

where Ψ_0 solves the Schrödinger equation with $\mathbf{Q} = 0$ [19]. Although the phase ϕ can be removed by a phase transformation, the phase shift, or phase shift variation, is an observable quantity that is phase (or gauge) invariant [8,20].

2.2 Light waves

The basic arguments that lead to the formulation of the magnetic model for light in moving media are the following. Let us consider the wave equation for light in a moving medium, which is the Lorentz transformation of the wave equation $[\nabla'^2 - (n^2/c^2)\partial_{t'}^2]\Psi = 0$ written in the comoving frame K' of the medium. The Lorentz transformations for the coordinates, frequencies and wave vectors between the laboratory frame K and the frame $K' = K_o$ comoving with the flow are

$$\begin{aligned} t_o &= \gamma\left(t - \frac{\mathbf{u}\cdot\mathbf{x}}{c^2}\right) \\ \mathbf{x}_o &= \mathbf{x} + \frac{\gamma-1}{u^2/c^2}\frac{(\mathbf{u}\cdot\mathbf{x})}{c}\frac{\mathbf{u}}{c} - \gamma\mathbf{u}\,t \\ \omega_o &= \gamma\,(\omega - \mathbf{u}\cdot\mathbf{k}) \\ \mathbf{k}_o &= \mathbf{k} + \frac{\gamma-1}{u^2/c^2}\frac{(\mathbf{k}\cdot\mathbf{x})}{c}\frac{\mathbf{u}}{c} - \gamma\,\frac{\omega}{c^2}\,\mathbf{u} \end{aligned} \tag{4}$$

but here, and in the following equations, we use an approximation in the lowest order in u/c. According to (4) the directions of the light ray in K and K_o are related by

$$\mathbf{e} = \frac{\mathbf{k}}{k} = \frac{\mathbf{k}_o + \omega_o\mathbf{u}/c^2}{(k_o^2 + 2\omega_o\mathbf{u}\cdot\mathbf{e}/c^2)^{1/2}} = \mathbf{e}_o - \mathbf{e}\frac{(\mathbf{u}\cdot\mathbf{e})}{nc} + \frac{\mathbf{u}}{nc}. \tag{5}$$

The wave equation reads as follows in the laboratory frame

$$\left(\nabla^2 - 2\frac{n^2-1}{c^2}\mathbf{u}\cdot\nabla\,\partial_t - \frac{n^2}{c^2}\partial_t^2 + O(u^2)\right)\Psi = 0. \quad (6)$$

Its validity is not restricted to a constant uniform flow $\mathbf{u}$. In fact, the wave equation (6) is valid for a nondispersive dielectric medium where both n and $\mathbf{u}$ vary in space and time, provided that they do not change significantly over the spatial scale of an optical wave length and over one optical cycle, respectively [10]. The mentioned wave equation can be derived, using special relativity, from the wave equation in the comoving frame of the medium conceiving it to be composed of small cells or drops. Each cell should be small enough such that n and the velocity profile $\mathbf{u}$ of the medium does not vary significantly, and each cell should be large compared to the wavelength of light. However, since in the present context we are dealing with effects of the AB type, where $\mathbf{Q}(\mathbf{x})$ does not depend on time, for our purposes it would be sufficient to assume here that the nonuniform flow $\mathbf{u}(\mathbf{x})$ is time independent.

In units of $\hbar = 1$, we seek for a solution of the type of (3), i.e.,

$$\Psi = e^{i\phi}\Psi_0 = e^{i\int(\mathbf{k}\cdot d\mathbf{x}-\omega\, dt)}\mathcal{A} \quad (7)$$

where $\mathbf{k}_0 = \mathbf{k}'$ and $\mathbf{k}$ are wave vectors, $\omega' = \omega_o = k_o c/n$ and ω the angular frequencies, and n the index of refraction, while Ψ_0 solves equation (6) with $\mathbf{u} = 0$. We substitute the ansatz corresponding to the last term of equation (7) into the wave equation (6), neglect the variation of the amplitude $\mathcal{A}$, and obtain the dispersion relation [10]

$$k^2 - \frac{n^2}{c^2}\left(\omega - \frac{n^2-1}{n^2}\mathbf{u}\cdot\mathbf{k}\right)^2 = 0. \quad (8)$$

The Hamiltonian of light rays H is equal to the frequency ω,

$$H = \omega = \frac{c}{n}k + \left(1 - \frac{1}{n^2}\right)\mathbf{u}\cdot\mathbf{k}. \quad (9)$$

and the Hamilton's equations are

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{k}}, \qquad \frac{d\mathbf{k}}{dt} = -\frac{\partial H}{\partial \mathbf{x}}. \quad (10)$$

The first of Hamilton's equations (10) provides the ray velocity $\mathbf{v} = d\mathbf{x}/dt$ that must coincide in the first order with the relativistic expression [21]

$$\begin{aligned}\mathbf{v} &= \frac{\mathbf{v}' + \mathbf{u} + (\gamma-1)\mathbf{u}(\mathbf{u}\cdot\mathbf{v}' + u^2)/u^2}{\gamma(1+\mathbf{u}\cdot\mathbf{v}'/c^2)}\\ &\simeq \mathbf{v}' + \mathbf{u} - \mathbf{v}'\frac{\mathbf{u}\cdot\mathbf{v}'}{c^2}\\ &= \frac{\mathbf{c}}{n} + \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{e}}{n^2}\mathbf{e}\end{aligned} \quad (11)$$

where $\mathbf{c}/n = (c/n)\,\mathbf{e}_o$ while

$$\gamma_v = \gamma_u\gamma_{v'}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}'}{c^2}\right),$$
$$\text{and}\quad \gamma = \gamma_u = (1 - u^2/c^2)^{-1/2}. \quad (12)$$

Thus, the Hamiltonian (9) leads to the light velocity

$$\mathbf{v} = \frac{\partial H}{\partial \mathbf{k}} = \frac{c}{n}\mathbf{e} + \left(1 - \frac{1}{n^2}\right)u = \frac{\mathbf{c}}{n} + \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{e}}{n^2}\mathbf{e} \quad (13)$$

in agreement with (11).

2.3 The Fresnel-Fizeau momentum and the wave equation for light

In agreement with special relativity, the velocity (13) is the vector version of the original Fresnel-Fizeau speed v of equation (1), where v is valid only for propagation in the direction of the flow. Thus, let us introduce the Fresnel-Fizeau momentum proportional to the velocity variation $(\Delta\mathbf{v})_{\mathbf{c}} = \mathbf{v} - \mathbf{c}/n$

$$\mathbf{Q}_{\mathbf{c}} = \frac{n^2}{c^2}\omega\left[\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{e}}{n^2}\mathbf{e}\right] = \frac{n^2}{c^2}\omega(\Delta\mathbf{v})_{\mathbf{c}}. \quad (14)$$

For the velocity variation $\Delta\mathbf{v} = \mathbf{v} - (c/n)\mathbf{e}$ the corresponding Fresnel-Fizeau momentum reads

$$\mathbf{Q} = \frac{\omega}{c^2}(n^2-1)\mathbf{u} = \frac{n^2}{c^2}\omega\Delta\mathbf{v}. \quad (15)$$

The difference between $\mathbf{Q}_{\mathbf{c}}$ and $\mathbf{Q}$ disappears in the scalar products $\mathbf{Q}_{\mathbf{c}}\cdot\mathbf{k} = \mathbf{Q}\cdot\mathbf{k}$.

Since $w^2 = \omega^2$, replacing $\mathbf{w}$ with the rhs of equation (49) and $\mathbf{k}\Psi$ with $-i\nabla\Psi$ the resulting dispersion relation translates [10] into the wave equation for light in slowly moving media

$$\left[-i\nabla + \frac{\omega}{c^2}(n^2-1)\mathbf{u}\right]^2\Psi = (-i\nabla + \mathbf{Q})^2\Psi = n^2\frac{\omega^2}{c^2}\Psi, \quad (16)$$

which is analogous to the Schrödinger equation of a charged matter wave in a magnetic field [22]. Either $\mathbf{Q}_{\mathbf{c}}$ or $\mathbf{Q}$ can be indifferently used to form the canonical momentum in the wave equation (16). Actually, the same wave equation (6) can be derived without reference to special relativity by taking into account the polarization of the moving medium [23].

Equations (2) and (16) are analogous wave equations. In equation (2) the momentum p is that of a material particle, while, if p is taken to be the momentum $\hbar k$ of light (in units of $\hbar = 1$), equation (2) becomes equation (16), $\mathbf{Q}$ being the corresponding appropriate interaction momentum for each case.

In the Fizeau experiment the observed fringes in the interference pattern for light in moving water was proportional to the phase variations related to the Fresnel-Fizeau term (14) or (15). In order to evidence this phase variation and for a closer analogy with the wave equation of the AB effects, save for the sign of $\mathbf{Q}$, we may write the solution Ψ of the wave equation (6) in a form analogous to solution (3), i.e., with phase

$$\begin{aligned}\Phi &= \phi + \int(\mathbf{k}_o\cdot d\mathbf{x} - \omega_o\, dt)\\ &= -\int\mathbf{Q}\cdot d\mathbf{x} + \int(\mathbf{k}_o\cdot d\mathbf{x} - \omega_o\, dt).\end{aligned} \quad (17)$$

For light propagating in the direction of the flow, the observable phase variation, corroborated by the Fizeau experiment and corresponding to the variation $\Delta\mathbf{u} = \mathbf{u}$ of the flow, is

$$|\Delta\Phi| = \int \mathbf{Q} \cdot d\mathbf{x} = \int \frac{\omega}{c^2}\,(n^2 - 1)\,u\,dx. \tag{18}$$

The above arguments support the idea of a magnetic model of light propagation in slowly moving media [2,10], while the vectorial form of the Fresnel-Fizeau term $\mathbf{Q}$ (14) or (15) corresponds to a velocity $\mathbf{v}$ in agreement with special relativity.

3 The interaction electromagnetic momentum for matter and light waves

In Section 2 we have pointed out that the wave equation (6) can be obtained either by means of special relativity or from the Maxwell equations for a moving fluid. Now we wish to establish the relation between the interaction momentum $\mathbf{Q}$ and the linear momentum of the em fields $\mathbf{P}_e$. In general, with T^M_{ik} the Maxwell stress-tensor, the covariant description of the em momentum (Ref. [21], Sect. 17.5) leads to the four-vector em momentum P^α_e expressed as

$$P^i_e\,c = \gamma \int (c\mathbf{g} + T^M_{ik}\beta^i) d^3\sigma$$
$$cP^0_e = \gamma \int (u_{em} - \mathbf{v} \cdot \mathbf{g}) d^3\sigma \tag{19}$$

where $\beta = v/c$, and the em energy and momentum are evaluated in a special frame $K^{(0)}$ moving with velocity $\mathbf{v}$ with respect to the laboratory frame. Different expressions for the momentum density $\mathbf{g}$ have been used in the literature (Ref. [21], Sect. 6-9), such as $\mathbf{g} \propto \mathbf{E} \times \mathbf{B}$, $\mathbf{g} \propto \mathbf{D} \times \mathbf{B}$, and $\mathbf{g} \propto \mathbf{E} \times \mathbf{H}$.

3.1 The em momentum in the effects of the AB type

Although with $\mathbf{Q} = (e/c)\mathbf{A}$ equation (2) describes the magnetic Aharonov-Bohm effect, the same equation can be used to describe many other quantum effects denoted generically as effects of the AB type. In fact, all the effects of the AB type discussed in the literature [3–8] can be described by equation (2), provided that the interaction momentum $\mathbf{Q}$ is related [6,7] to $\mathbf{P}_e$, the momentum of the em fields.

In the mentioned effects of the AB type, $\mathbf{Q}$ has been expressed as [3–7]

$$\mathbf{Q} = \pm\mathbf{P}_e = \pm\,\frac{1}{4\pi c}\int (\mathbf{E} \times \mathbf{B}) d^3\mathbf{x}'$$
$$\text{or} \qquad \pm\,\frac{1}{4\pi c}\int (\mathbf{D} \times \mathbf{B}) d^3\mathbf{x}' \tag{20}$$

where $\mathbf{E}$ is the electric field, $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}_{pol}$ the electric displacement, $\mathbf{P}_{pol}$ the electric polarization and $\mathbf{B}$ the magnetic field.

If $\mathbf{Q}$ is thought of as describing a moving fluid or a flow $\mathbf{u}$, the particles or matter waves propagate through this moving em fluid. In the terminology of fluid dynamics, the interaction momentum $\mathbf{Q}(\mathbf{x})$ is proportional to the flow $\mathbf{u}(\mathbf{x})$ arising from the em interaction and the quantity $\nabla \times \mathbf{Q} \propto \nabla \times \mathbf{u}$ represents its vorticity [6].

The AB term $\mathbf{Q} = (e/c)\mathbf{A}$ is obtained by taking $\mathbf{E}$ in equation (20) to be the electric field of the charge and $\mathbf{B}$ to be the magnetic field of the solenoid. A general proof that this result holds in the *natural* Coulomb gauge, is given by Boyer [24], Zhu and Henneberger [25], and Spavieri [26]. Actually, the observable quantity is not the phase but the phase shift or the phase shift variation $\Delta\Phi$ [20]. What is physically meaningful is the variation of $\mathbf{P}_e$ related to $\Delta\Phi$, as in equation (18) for the Fizeau experiment. For example, to test the AB effect with a solenoid, a tapering iron whisker was used [27] so that its magnetic flux varies along its length. The AB relative shift $\Delta\phi_{AB}$ was observable by comparing the relative position of the sets of fringes displaced, or tilted, by the varying magnetic flux of different segments of the whisker. Thus, $\Delta\phi_{AB}$ refers to fringe displacements arising for example from comparing the interference pattern corresponding to $\mathbf{Q}(\mathbf{u}) = \mathbf{P}_e = (e/c)\mathbf{A}$ with the interference pattern corresponding to $\mathbf{Q}(\mathbf{u} = 0) = \mathbf{P}_{oe} = (e/c)\mathbf{A} = 0$.

We consider it convenient to recall how in the other effects of the AB type $\mathbf{Q}$ is related to $\mathbf{P}_e$ by (20). The idea is to emphasize that the magnetic model of light propagation in moving media is not linked exclusively to the AB effect.

In the case of the AC effect [4], for particles possessing a magnetic dipole moment $\mathbf{m}$ and moving in the presence of a field $\mathbf{E}$, we have $\mathbf{Q} = \mathbf{m} \times \mathbf{E}/c = -\mathbf{P}_e = -(4\pi c)^{-1} \int (\mathbf{E} \times \mathbf{B}) d^3\mathbf{x}'$ [28]. For an electric dipole $\mathbf{d}$ in a magnetic field $\mathbf{B}$, as in the He-McKellar, Wilkens, Tkachuk [5], and Spavieri [5,6] effects, equation (20) yields $\mathbf{Q} = (\mathbf{d} \cdot \nabla)\mathbf{A}/c = \mathbf{P}_e = (4\pi c)^{-1} \int (\mathbf{E} \times \mathbf{B}) d^3\mathbf{x}'$.

For particles possessing a magnetic dipole moment $\mathbf{m}$ and interacting with the field $\mathbf{D}$ of a distribution of electric dipoles as in the quantum effect for a magnetic dipole described in reference [7], considerations on Maxwell's duality lead to an interaction term which is dual of that of the electric dipole and to the momentum of the em field as given by the last term of equation (20). In this case, the interaction momentum reads $\mathbf{Q} = -c^{-1}(\mathbf{m} \cdot \nabla)\mathbf{A}_d = \mathbf{P}_e = (4\pi c)^{-1} \int (\mathbf{D} \times \mathbf{B}) d^3\mathbf{x}'$, where $\mathbf{A}_d$ is the vector potential due to the electric dipole distribution, dual of the usual vector potential $\mathbf{A}$ of a magnetic dipole distribution.

In conclusion, the wave equation (2) with the term $\mathbf{Q}$ of equation (20), describes in a unitary way all the effects of the AB type, regardless of the type of particles and em source distribution.

The $\mathbf{P}_e$ of the effects of the AB type coincides with the em momentum $\mathbf{P}_e = \int \mathbf{g}\, d^3\mathbf{x}$ of equation (19) with the em energy and momentum evaluated in the laboratory frame ($\mathbf{v} = 0$). Actually, $\mathbf{P}_e$ is calculated for static configuration, i.e., when also the particle velocity is $\mathbf{v} = 0$. Therefore, in these effects there is em momentum (and em flow $\mathbf{u}$) even when the particle is at rest [28]. In the analogy with fluid dynamics mentioned above, it is this interaction em momentum that determines and represents the em flow $\mathbf{u}$ where the matter waves of effects of the AB type propagate.

3.2 The em momentum for light in slowly moving media

So far, we have shown that the wave equation (16) for light propagating in a moving medium is formally identical to equation (2) of the effects of the AB type. To corroborate the physical equivalence of the two wave equations we have to relate the Fresnel-Fizeau term $\mathbf{Q}$ to the variation of the em momentum $\mathbf{P}_e$ of the light ray calculated using the general expression (19).

In the case of a light wave in a moving fluid, the flow $\mathbf{u}$ is created by external mechanical agents (forces or pressure) that make the fluid move in the pipe or container where it flows. An interaction momentum and an em flow arises only when both the fluid moves with velocity $\mathbf{u}$ and the effective fields of the wave propagating in the fluid interact with the mechanical flow to produce interaction em fields such as the polarization $\mathbf{P}_{pol} = 4\pi^{-1}(n^2-1)\mathbf{E}$. The momentum $\mathbf{Q}$ vanishes if there is no light wave or if the mechanical flow vanishes ($\mathbf{u} = 0$).

In the Fizeau experiment a light wave with momentum $\mathbf{k}_0$, frequency ω_0 and speed ω_0/k_0, originally in a medium at rest, propagates into a moving medium. In the case of refraction, for the propagation of light at a plane boundary between two media that possess different optical properties, the propagation vector or momentum $\mathbf{k}_0$ is modified to $\mathbf{k}$. In the previous section the interaction momentum $\mathbf{Q}(\mathbf{u})$ has been related to the velocity variation of light $\Delta\mathbf{v}$ due to the flow. According to Panofsky and Phillips [23], this velocity variation is linked to the polarization current that corresponds to the motion of dipoles that are affected by the velocity of the medium. The resulting wave that depends on $\mathbf{u}$ combines with the original light wave, which acquires the overall phase velocity v.

Thus, as done in the next Sections, we have to relate explicitly $\mathbf{Q}(\mathbf{u})$ with the variation of the em momentum $\mathbf{P}_e$ due to the flow and with the polarization of the medium.

4 The em momentum of light in a moving medium

The energy density of em waves propagating in a dielectric medium is [21] $u_{em} = (1/8\pi)(\varepsilon\mathbf{E}_o^2 + \mathbf{B}_o^2) = (\varepsilon/4\pi)\mathbf{E}_o^2$ and that of the energy flow is $\mathbf{S} = \mathbf{g}c^2 = (c/4\pi)\mathbf{E}_o{\times}\mathbf{B}_o^* = (c/4\pi)\sqrt{\varepsilon}\,\mathbf{E}_o^2\ \mathbf{e}_o$, while the speed of the energy flow is $v = |\,\mathbf{S}|/u_{em} = c/\sqrt{\varepsilon} = c/n$.

The standard classical-quantum correspondence ($\hbar = 1$)

$$\int u_{em}\, d^3\mathbf{x}' = \frac{1}{4\pi}\int \varepsilon\,(\mathbf{E}_o^2)\, d^3\mathbf{x}' \to n^2\omega_0$$
$$c^{-1}\int \mathbf{g}\, d^3\mathbf{x}' = \frac{c^{-1}}{4\pi}\sqrt{\varepsilon}\,\mathbf{e}_o\int (\mathbf{E}_o^2)\, d^3\mathbf{x}' \to \mathbf{k}_0 \qquad (21)$$

holds for the energy $\varepsilon\,\omega_o$ and the momentum $\mathbf{k}_o$.

4.1 The momentum Q, the em momentum $\mathbf{P}_e$ and the mass of the em fields

To describe the em momentum of light in our moving medium we use the following momentum density and Maxwell tensor (Ref. [21], Sect. 6-9)

$$\mathbf{g} = \frac{1}{4\pi c}(\mathbf{E}\times\mathbf{H})$$
$$T_{\alpha\beta} = \frac{1}{4\pi}(E_\alpha D_\alpha + H_\alpha B_\alpha - \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})\delta_{\alpha\beta} \qquad (22)$$

where, in our case, we have $\mathbf{H} = \mathbf{B}$.

There are different possibilities for the choice of the special frame $K^{(0)}$ and different choices of the frame $K^{(0)}$ lead to different 4-vectors (Ref. [21], Sect. 17.5). A possible choice for $K^{(0)}$ consists of defining the em energy and momentum in the inertial frame $K' = K_o$ comoving with the fluid

$$E_{oe} = \frac{1}{8\pi c}\int [\mathbf{D}_o\cdot\mathbf{E}_o + \mathbf{B}_o\cdot\mathbf{H}_o] d^3x' = n^2\omega_o$$
$$\mathbf{P}_{oe} = \frac{1}{4\pi c}\int \mathbf{E}_o{\times}\mathbf{H}_o d^3x' = \mathbf{k}_o \qquad (23)$$

to obtain from (22) and (19)

$$\mathbf{P}_e = \gamma\frac{1}{4\pi c}\int \Big[\mathbf{E}\times\mathbf{H} + \Big(\frac{\mathbf{u}}{c}\cdot\mathbf{E}\Big)\mathbf{D} + \Big(\frac{\mathbf{u}}{c}\cdot\mathbf{H}\Big)\mathbf{B} - \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})\frac{\mathbf{u}}{c}\Big]\, d^3x' \qquad (24)$$

The correspondence $\mathbf{P}_{oe} = \mathbf{k}_o$ of equations (21) and (23) holds only in the rest frame of the fluid since in general $\mathbf{P}_e \neq \mathbf{k}$.

Let us now derive, in first order in u/c, the link between $\mathbf{P}_e$ and $\mathbf{P}_{oe}$ using the relations (23) and (24). Expressing the fields $\mathbf{E}$ and $\mathbf{B}$ in terms of $\mathbf{E}_o$ and $\mathbf{B}_o$ and $\mathbf{D}$ and $\mathbf{H}$ in terms of $\mathbf{D}_o$ and $\mathbf{H}_o$ [29]

$$\mathbf{E}\times\mathbf{H} = \Big(\mathbf{E}_o - \frac{\mathbf{u}}{c}\times\mathbf{B}_o\Big)\times\Big(\mathbf{H}_o + \frac{\mathbf{u}}{c}\times\mathbf{D}_o\Big) \qquad (25)$$
$$= \mathbf{E}_o{\times}\mathbf{H}_o + (\mathbf{E}_o{\cdot}\mathbf{D}_o + \mathbf{B_o}\cdot\mathbf{H}_o)\frac{\mathbf{u}}{c} - \Big(\frac{\mathbf{u}}{c}\cdot\mathbf{D}_o\Big)\mathbf{E}_o - \Big(\frac{\mathbf{u}}{c}\cdot\mathbf{H}_o\Big)\mathbf{B}_o$$

$$(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})\frac{\mathbf{u}}{c} \simeq (\mathbf{D}_o\cdot\mathbf{E}_o + \mathbf{B}_o\cdot\mathbf{H}_o)\frac{\mathbf{u}}{c}. \qquad (26)$$

Substituting (25) and (26) into (24)

$$\begin{aligned}\mathbf{P}_e = \frac{1}{4\pi c}\int \Big\{ &[\mathbf{E}_o \times \mathbf{H}_o + (\mathbf{D}_o\cdot\mathbf{E}_o + \mathbf{B}_o\cdot\mathbf{H}_o)\frac{\mathbf{u}}{c} \\ &-\left(\frac{\mathbf{u}}{c}\cdot\mathbf{E}_o\right)\mathbf{D}_o - \left(\frac{\mathbf{u}}{c}\cdot\mathbf{H}_o\right)\mathbf{B}_o \\ &+\left(\frac{\mathbf{u}}{c}\cdot\mathbf{E}_o\right)\mathbf{D}_o + \left(\frac{\mathbf{u}}{c}\cdot\mathbf{H}_o\right)\mathbf{B}_o \\ &-\frac{1}{2}(\mathbf{D}_o\cdot\mathbf{E}_o + \mathbf{B}_o\cdot\mathbf{H}_o)\frac{\mathbf{u}}{c}\Big\} d^3\mathbf{x}' \\ = &\frac{1}{4\pi c}\int (\mathbf{E}_o\times\mathbf{H}_o) d^3\mathbf{x}' \\ &+\frac{1}{4\pi c}\int \left[\frac{1}{2}(\mathbf{D}_o\cdot\mathbf{E}_o + \mathbf{B}_o\cdot\mathbf{H}_o)\frac{\mathbf{u}}{c}\right] d^3\mathbf{x}' \\ = &\mathbf{P}_{oe} + \frac{\omega}{c^2}n^2\mathbf{u}. \end{aligned} \tag{27}$$

We should recover result (27) even with a different choice of the special of frame $K^{(0)}$. The natural choice for $K^{(0)}$ is the rest frame in which the momentum $\int(\mathbf{g})d^3\sigma$ vanishes, i.e., the frame comoving with the light ray. In this frame, the em rest energy is

$$\begin{aligned} m_e c^2 &= \frac{1}{8\pi}\int [\mathbf{E}^{(0)}\cdot\mathbf{D}^{(0)} + \mathbf{B}^{(0)}\cdot\mathbf{H}^{(0)}] d^3\sigma \\ &= \frac{n^2}{4\pi}\int (\mathbf{E}^{(0)})^2 d^3\sigma. \end{aligned} \tag{28}$$

as if the em mass m_e of the fields were nonvanishing. In the frame comoving with the medium the em energy is

$$\gamma_{c/n} m_e c^2 = n^2 \omega_o,$$

and

$$m_e = n^2 \frac{\omega_o}{\gamma_{c/n} c^2} = n(n^2-1)^{1/2}\frac{\omega_o}{c^2}. \tag{29}$$

The em momentum in the laboratory frame and in the frame comoving with the fluid read respectively

$$\begin{aligned} \mathbf{P}_e &= \gamma_v\, m_e \mathbf{v} = \frac{\gamma_v}{\gamma_{c/n}}\, n^2 \frac{\omega_o}{c^2}\mathbf{v} \\ \mathbf{P}_{oe} &= \gamma_{c/n} m_e \frac{\mathbf{c}}{n} = n^2\frac{\omega_o}{c^2}\frac{\mathbf{c}}{n}. \end{aligned} \tag{30}$$

With the help of $\gamma_v = \gamma_u \gamma_{c/n}(1 + \mathbf{u}\cdot\mathbf{c}/nc)$ from (12) we obtain

$$\mathbf{P}_e - \mathbf{P}_{oe} = \frac{\omega}{c^2}n^2\mathbf{u} \tag{31}$$

in agreement with (27). We see from (27) or (31) that the variation $\mathbf{P}_e - \mathbf{P}_{oe}$ provides only the leading term of the Fresnel-Fizeau momentum.

4.2 Links of Q with $\mathbf{P_e}$ and the em mass

From (30) and (12) we find the relationship

$$\mathbf{Q_c} = \mathbf{P}_e - \frac{\gamma_{v_\phi}}{\gamma_{c/n}}\mathbf{P}_{oe} = \frac{\omega}{c^2}n^2\left[\mathbf{u} - \frac{(\mathbf{u}\cdot\mathbf{e})\mathbf{e}}{n^2}\right] = \frac{\omega n^2}{c^2}(\Delta\mathbf{v})_\mathbf{c}. \tag{32}$$

The corresponding relation for $\mathbf{Q}$ is

$$\mathbf{Q} = \mathbf{P}_e - \frac{\gamma_v}{\gamma_{c/n}}P_{oe}\mathbf{e} = \frac{\omega n^2}{c^2}\left(1 - \frac{1}{n^2}\right)\mathbf{u} = \frac{\omega n^2}{c^2}\Delta\mathbf{v}. \tag{33}$$

The interpretation of expressions (32) and (33) in terms of nonconservation of simultaneity and the variation of the index of refraction is given elsewhere.

A simple interpretation of the interaction momentum $\mathbf{Q}$ (or $\mathbf{Q_c}$) arises by introducing the mass of the em fields $m = \gamma_v m_e$ in the laboratory frame. With this mass equations (32) and (33) yield

$$\mathbf{Q_c} = m\mathbf{v} - m\frac{\mathbf{c}}{n} = m(\Delta\mathbf{v})_\mathbf{c} \tag{34}$$

and

$$\mathbf{Q} = m\mathbf{v} - m\frac{c}{n}\mathbf{e} = m\Delta\mathbf{v}. \tag{35}$$

Equations (32), (33), (34) and (35) link the Fresnel-Fizeau term $\mathbf{Q}$ (or $\mathbf{Q_c}$) to variations of the em momentum (19). However, these links of the Fresnel-Fizeau term with the em momentum $\mathbf{P}_e$ do not express $\mathbf{Q}$ as a net variation of $\mathbf{P}_e$. The fact is that $\mathbf{P}_e$ is the total em momentum, i.e.: momentum carried by light waves + interaction momentum associated with the flowing medium. The em fields of light polarize the medium while the motion of the resulting dipoles are affected by the velocity of the medium producing the interaction momentum $\mathbf{Q}$. In fact, according to the considerations at the end of Section 3, $\mathbf{Q}$ is an interaction momentum that is linked to the interaction of the em waves with the flowing medium only, i.e., the interaction em momentum due to polarization associated with the flow. This relation with the polarization will be established rigorously in Section 5 by taking into account the variation of the refractive index.

5 The Fresnel-Fizeau momentum and the interaction em momentum

One of the approaches to $\mathbf{Q}$ (described elsewhere) consists of varying the index of refraction. This approach offers a new insight in the interpretation of the Fresnel-Fizeau momentum and hints to look for $\mathbf{Q}$ in terms of the net variation of the refractive index. We pursue this approach considering the general case of a flow with relativistic velocity.

A completely relativistic theory of light propagation in moving nondispersive media has been outlined in reference [10] and is briefly recalled here for convenience of the reader. We assume that the refractive index n and the flow $\mathbf{u}$ vary gradually over one optical wave length and one optical cycle, respectively. Otherwise, the medium velocity $\mathbf{u}$ is arbitrary.

In a frame ($K' \equiv K_o$) comoving with the medium the em field-strength tensor $F_{o\mu v}$ obeys the wave equation

$$\left(\nabla_o^2 - \frac{n^2}{c^2}\frac{\partial^2}{\partial t_o^2}\right)F_{o\mu v} = 0. \tag{36}$$

Introducing the four-vector field of the medium flow

$$u^{\upsilon} = \gamma\left(1, \frac{\mathbf{u}}{c}\right), \qquad u_{\mu} = \gamma\left(1, -\frac{\mathbf{u}}{c}\right) \tag{37}$$

and transforming (36) to the laboratory frame we arrive at [10]

$$[\partial_{\alpha}\partial^{\alpha} + (n^2-1)(u^{\alpha}\partial_{\alpha})^2]F_{\mu\upsilon} = 0. \tag{38}$$

With the ansatz

$$S = \int(\mathbf{k}\cdot d\mathbf{x} - \omega dt) = -\int k_{\nu}dx^{\nu} \tag{39}$$

Leonhardt and Piwnicki obtain the dispersion relation

$$\omega^2 - c^2k^2 + (n^2-1)\gamma^2(\omega - \mathbf{u}\cdot\mathbf{k})^2 = 0. \tag{40}$$

Solving equation (40) for $\omega = H$ yields a Hamiltonian that may be expressed as

$$H = k\,h(\zeta), \qquad \zeta = \mathbf{u}\cdot\mathbf{e}. \tag{41}$$

The remarkable structure of H of equation (41) is that it implies a velocity vector independent of the wave number k.

The rest frame dispersion relation $n^2\omega_o^2 - c^2k_o^2 = 0$, corresponding to the wave equation (36) with the ansatz (39) written in K_o, is

$$\omega_o^2 - c^2k_o^2 + (n^2-1)\omega_o^2 = 0. \tag{42}$$

Relation (40) reduces to (42) with the Lorentz scalar

$$\omega_o^2 - c^2k_o^2 = \omega^2 - c^2k^2 \quad \text{and} \quad \omega_o = \gamma(\omega - \mathbf{u}\cdot\mathbf{k}) \tag{43}$$

With the last equation of (43) we write (40) as

$$\omega^2 - c^2k^2 + (n^2-1)\omega_o^2 = 0. \tag{44}$$

The term $(n^2-1)\omega_o^2$ in (42) and (44) represents the variation of the dispersion relations due to the change of refractive index from $n = 1$ to n. We associate to this variation the em energy E_{oi} and em mass m_i

$$E_{oi} = \gamma_{c/n}m_ic^2 = (n^2-1)\omega_o, \qquad m_i = \frac{(n^2-1)}{n^2}m_e, \tag{45}$$

which vanish when $n \to 1$. In the special frame $K^{(0)}$ the em energy is $E_i^{(0)} = m_ic^2$. In the comoving frame of the medium, $E_{oi} = E_{oe}(n) - E_{oe}(n=1)$ represents the energy due to polarization acquired by the system, which arises from the interaction of the em wave with the medium. It is part of the total energy $E_{oe} = n^2\omega_o$, being $E_{oe}(n=1) = \omega_o$ the energy of the em wave in the absence of medium. Since the polarization of the medium is proportional to (n^2-1), we denote E_{oi} in (45) as the interaction polarization energy. The speed of the interaction energy flow is $v = |c^2\mathbf{P}_{oi}|/E_{oi} = c/n$.

The invariant em energy-momentum relation is

$$E_i^2 - c^2\mathbf{P}_i\cdot\mathbf{P}_i = m_i^2c^4. \tag{46}$$

From (45) we obtain the interaction polarization em momenta

$$\mathbf{P}_i = \gamma_v m_i\mathbf{v} = (n^2-1)\frac{\gamma_v}{\gamma_{c/n}}\frac{\omega_o}{c^2}\mathbf{v},$$
$$\mathbf{P}_{oi} = \gamma_{c/n}m_i\frac{\mathbf{c}}{n} = (n^2-1)\frac{\omega_o}{c^2}\frac{\mathbf{c}}{n}. \tag{47}$$

With $\gamma_v = \gamma_{c/n}\gamma(1 + \mathbf{u}\cdot\mathbf{e}_o/cn)$ from (12) equation (47) yields

$$\begin{aligned}\mathbf{P}_i(\mathbf{u}) - \mathbf{P}_{oi} &= (n^2-1)\frac{\omega_o}{c^2}\left(\frac{\gamma_v}{\gamma_{c/n}}\mathbf{v} - \frac{\mathbf{c}}{n}\right)\\ &= (n^2-1)\frac{\omega_o}{c^2}\left[\gamma(1+\mathbf{u}\cdot\mathbf{e}_o/cn)\mathbf{v} - \frac{\mathbf{c}}{n}\right]\\ &\simeq \frac{\omega}{c^2}(n^2-1)\mathbf{u}.\end{aligned} \tag{48}$$

The term at the rhs of equation (48) represents the exact relativistic variation $\mathbf{P}_i(\mathbf{u}) - \mathbf{P}_{oi}$ of the interaction polarization em momentum. The last term of equation (48), the variation $\mathbf{P}_i(\mathbf{u}) - \mathbf{P}_{oi}$ in first order in u/c, is the Fresnel-Fizeau momentum $\mathbf{Q}$ as given by equation (15).

Thus, the Fresnel-Fizeau momentum $\mathbf{Q}$ is given by the variation of the interaction polarization em momentum $\mathbf{P}_i$ due the flow $\mathbf{u}$, i.e., is the dragged interaction em momentum. This interpretation in terms of the drag of the interaction polarization momentum agrees with the one given by Panofsky and Phillips [23] mentioned at the end of Section 4.

5.1 A Lorentz-type equation of motion for the em momentum

After having derived the Fresnel-Fizeau momentum $\mathbf{Q}$ as the variation of the interaction polarization em momentum, we seek for a Lorentz-type equation of motion for the em momentum. To this end Leonhardt and Piwnicki [10] consider a reparametrization of the ray trajectory in terms of a rescaled vector $\mathbf{w} \equiv k\mathbf{v}$ that yields

$$\mathbf{w} \equiv k\mathbf{v} = \frac{c}{n}\mathbf{k} + k\left(1 - \frac{1}{n^2}\right)\mathbf{u} = \frac{c}{n}(\mathbf{k} + \mathbf{Q}). \tag{49}$$

In order to interpret the meaning of this rescaled vector, we introduce here the rescaled momentum

$$\frac{n}{c}\mathbf{w} \equiv \frac{n}{c}k\mathbf{v} = \mathbf{k} + \frac{c}{n}k\left(1 - \frac{1}{n^2}\right)\mathbf{u} = \mathbf{k} + \mathbf{Q}, \tag{50}$$

which will be interpreted in terms of the em momentum. For the rescaled momentum $k(n/c)\mathbf{v}$, the relation $d\mathbf{u}/dt = (\mathbf{v}\cdot\nabla)\mathbf{u}$ and Hamilton's equations (10) lead to the Lorentz-type equation of motion

$$\frac{d}{dt}\left(\frac{n}{c}\mathbf{w}\right) = \frac{d}{dt}\left(\frac{nv}{c}\mathbf{k}\right) = (\nabla\times\mathbf{Q})\times\mathbf{v}, \tag{51}$$

where $\mathbf{Q} \to \mathbf{A}$ plays the role of a magnetic vector potential.

To interpret the rescaled quantity $\mathbf{w}n/c$ appearing in equations (50) and (51) we notice that, among the different possibilities for the choice of the special frame $K^{(0)}$ (Ref. [21], Sect. 17.5), a possible choice for $K^{(0)}$ consists of defining the em energy and momentum in the inertial frame K of the laboratory. Indeed, with $\mathbf{P}_{ek} = k\mathbf{e}$ equation (51) becomes

$$\frac{d}{dt}\left(\frac{n}{c}\mathbf{w}\right) = \frac{d}{dt}\left(\frac{nv}{c}\mathbf{P}_{ek}\right) = (\nabla \times \mathbf{Q})\times\mathbf{v}, \quad (52)$$

implying that the rescaled quantity $\mathbf{w}n/c \equiv k\mathbf{v}n/c$ corresponds to the rescaled em momentum $\mathbf{P}_{ek}v/(c/n)$. Thus, equation (52) represents a Lorentz-type equation of motion involving the em momentum.

In closing, the momenta $\mathbf{Q}$ for the effects of the AB type (14) and $\mathbf{Q}$ for light in moving media, have the same physical origin since both are related to the interaction em momentum (19), validating the arguments that lead to the formulation of the magnetic model of light for moving media.

6 Fizeau-Aharonov-Bohm effects in astrophysics

The AB type effects involve particles such as electrons, magnetic and electric dipoles [6,7], ionized atoms or molecules [8], etc. Given the analogy implied by the magnetic model of light, in a way the Fizeau experiment is a kind of AB effect for light particles, or photons. However, in the Fizeau experiment the flow $\mathbf{u}$ (that takes on the role of the vector potential $\mathbf{A}$) is uniform, while in the AB effect $\mathbf{A}$, or $\mathbf{Q}$, is not uniform and we have

$$\mathbf{Q} = \frac{e}{c}\mathbf{A} = |\mathbf{L}|\nabla\theta = \frac{L}{r}\widehat{\theta}, \quad (53)$$

where, in cylindrical coordinates $r^2 = x^2 + y^2$and $\mathbf{L} = \mathbf{r} \times \mathbf{Q}$ represents the constant em angular momentum [6].

We wish to discuss here in a very qualitative and tentative way the possibility of realizing an experiment of the Fizeau-AB type for photons. In order for a Fizeau type experiment to reproduce the conditions of the AB effect, the flow $\mathbf{u}$ must possess the same dependence on r and θ given by (53), so that $\nabla \times \mathbf{Q} = 0$ in equation (52). A fluid vortex with these characteristics may probably be reproduced in a laboratory. However, seeking for a physical environment where condition (53) is naturally fulfilled, we consider here rotating cosmic objects at the astrophysical scale, such as rotating galaxies, binary stars, rotating neutron stars, pulsars, etc. During the formation of a rotating cosmic object, some matter δm (dust, gas, radiation, interstellar matter, etc.) may be ejected from the object or trapped by the gravitational force [30]. This matter may be rotating in the periphery of the object with tangential speed u, forming a flow $\mathbf{u}(\mathbf{r})$ with an average index of refraction n.

The velocity distribution $\mathbf{u}(\mathbf{r})$ must be consistent with the conservation of the mechanical angular momentum. If I is the moment of inertia and $\dot{\theta}$ the angular velocity,

$$I\,\dot{\theta} = \delta m\, r^2 \frac{u}{r} = \delta m\, r\, u = const. \quad (54)$$

Thus, equation (54) implies that the flow $\mathbf{u}(\mathbf{r}) \propto \widehat{\theta}/r$ possesses physical characteristics analogous to that of the em flow $\mathbf{Q}$ (53) of effects of the AB type.

However, the dependence $\mathbf{u} = \mathbf{u}(\mathbf{r})$ implied by equation (54) is generally altered by the effect of gravitation of nearby objects, dark matter, etc. Moreover, angular momentum is not always constant: gas steadily radiates away energy so that angular momentum must somehow be reduced, from the high values associated with distended hydrogen clouds moving in differential rotation about the Galactic center, to the low values observed in stars [31]. Nevertheless, even for cosmic objects with complex structure such as galaxies, the complicated dependence $\mathbf{u} = \mathbf{u}(\mathbf{r})$ possesses regions of $\Delta\mathbf{r}$ where the $1/r$ dependence implied by equation (54) is satisfied [32].

Let us consider a distant source S, such as a bright star, and an observer O, eventually located on the Earth. Let the rotating cosmic object be placed somewhere between S and O, with its axis of rotation perpendicular to SO. If the dimensions of the rotating cosmic object are small with respect to the distance SO, the rotating cosmic object with its flow $\mathbf{u}(\mathbf{r})$ can be described as a very small rotating disk. Photons or light waves corresponding to rays coming from the source S, passing through the periphery of this rotating cosmic object will be phase-shifted in complete analogy with electron matter waves of the AB effect. In principle it should be possible to create interference patterns using beams of light passing near the centre of the rotating cosmic object and through the flow $\mathbf{u}(\mathbf{r})$ at the periphery. Most likely, some of these beams need to be reflected by a distant mirror in order to be made to converge to the location of the observer O.

The quantity measured, the phase shift $\Delta\phi$ ($\Delta\phi = 2\pi L$, [6]), is related to L and, being

$$\nabla \times \mathbf{Q} = \mathbf{L}\frac{\delta(r)}{r} \quad (55)$$

it follows from (55) and (15) that the vorticity of the flow of the rotating cosmic object is

$$\nabla \times \mathbf{u} = \frac{c^2}{\omega(n^2-1)}\mathbf{L}\frac{\delta(r)}{r} = \frac{c^2}{\omega(n^2-1)}\frac{\Delta\phi}{2\pi}\widehat{\mathbf{L}}\frac{\delta(r)}{r}. \quad (56)$$

Thus, physical quantities such as the *vortex strength* $\oint(\nabla \times \mathbf{u}) \cdot d\mathbf{S}$ of the flow of the cosmic object, which is linked to the index of refraction by equation (15) and represents a gauge of the *vorticity* of the flow of the rotating cosmic object, can be measured and is given by

$$\oint(\nabla \times \mathbf{u}) \cdot d\mathbf{S} = 2\pi\frac{c^2}{\omega(n^2-1)}L = \frac{c^2}{\omega(n^2-1)}\,\Delta\phi. \quad (57)$$

In most cases, the cosmic object will be composed of a hard opaque core of radius R and, at its periphery for $r \geq R$, it will be surrounded by the flow $\mathbf{u}(\mathbf{r})$. The speed of the flow generally coincides with that of the core at $r = R$, i.e., $u(r = R) = u_0$ = tangential speed of the core. In this case, we have

$$\oint (\nabla \times \mathbf{u}) \cdot d\mathbf{S} = \oint \mathbf{u} \cdot d\mathbf{x} = 2\pi R u_0. \tag{58}$$

Generally, the quantities R and u_0 can be measured independently through astrophysical observations. Thus, from equations (58) and (57) we derive a rough, tentative theoretical expression for the phase shift associated to this rotating cosmic object

$$\Delta\phi = 2\pi R u_0 \frac{\omega(n^2 - 1)}{c^2}. \tag{59}$$

Even for a rotating cosmic object made of dark matter the phase shift (59) should be observable and its detection may be used to reveal the existence of such a dark matter. If the dependence $\mathbf{u} = \mathbf{u}(\mathbf{r})$ does not satisfy equation (54) but is known, the discrepancy with prediction (59) may be calculated theoretically so that the result of observation may still be useful to determine rotational parameters or physical properties of cosmic objects.

Our suggestion is tentative and qualitative because we are unable to discuss in detail in this paper technical aspects of the interferometry necessary for the observation of this effect. We simply point out that the AB phase shift $\Delta\phi$ is a phase invariant quantity [20] and thus, as such, should be observable (see also the discussion on phase invariance of Ref. [8]). Considering the established analogy between matter waves and light waves, we may conclude that also the phase shift $\Delta\phi$ for light of equation (59) is an observable quantity. Although the details about technical aspects of the interferometry for the observation of an astrophysical Fizeau-Aharonov-Bohm effect are left to a future paper, rotating cosmic objects suitable for this effect are out there, ready and waiting for potential observations.

7 Conclusions

The recent magnetic model of light propagation in moving media [2,10] is supported by the equivalence of equations (2) and (16) for matter and light waves. In our approach we adhere to the standard interpretation of the effects of the AB type and assume that there is a physical analogy between the propagation of matter waves in an em flow $\mathbf{Q}$ and the propagation of light waves in a flow $\mathbf{u}$. In both equations the em flow corresponds to an interaction momentum $\mathbf{Q}$ that modifies the original wave equation and influences the phase of the wave function Ψ.

In the effects of the AB type, regardless of the distribution of the em sources and of the type of particles involved, the interaction momentum $\mathbf{Q}$ is related to the momentum of the interaction em fields $\mathbf{P}_e$ (19) while the observable quantity, the phase shift, is related to the variations of $\mathbf{P}_e$. Calculation of the interaction momentum $\mathbf{Q}$ for the a light wave propagating in a slowly moving medium leads to the Fresnel-Fizeau term, which is related to the em momentum (19) and is linked to the variation of the light ray em momentum $\mathbf{P}_e$ due to the flow. Finally, in equation (48) $\mathbf{Q}$ is derived as the net variation of the interaction em momentum $\mathbf{P}_i$ of polarization due the flow $\mathbf{u}$, i.e., the dragged interaction em momentum.

Hence, the interaction momenta of the effects of the AB type and of light in moving media have the same physical origin. When, by virtue of the classical-quantum correspondence the Fresnel-Fizeau em momentum $\mathbf{Q}$ is substituted for the corresponding momentum in the matter wave equation (2), it yields the equation (16) for light waves, confirming the assumed physical equivalence or correspondence between the two wave equations.

This result corroborates the magnetic model of light propagation in moving media and lends support to its several connections and applications to other fields of physics. Moreover, it contributes to point out the important role played by the em momentum, a role that has been only seldom and occasionally recognized or emphasized by physicists. Thus, for decades it has been not obvious that in effects of the AB type the interaction momentum $\mathbf{Q}$ had to do, or better, coincided with the interaction em momentum. However, the approach based on the em momentum in dealing with the classical and quantum wave equations has led to the derivation of the Fresnel-Fizeau momentum, to a corroboration of the magnetic model of light, to a unitary vision of the em interaction in all these effects [6] and to the discovery of new quantum effects [5–8], some of which were considered physically impossible [8].

In effects of the AB type the Lagrangian (and corresponding quantum Hamiltonian) of a particle possessing em properties and moving with velocity $\mathbf{v}$ contains an interaction energy of the type $\mathbf{v} \cdot \mathbf{Q}$ which can be constructed in general by substituting for $\pm\mathbf{Q}$ the interaction em momentum [7,33]. Obviously, particles with different em properties, such as electrons, em dipoles and photons, possess different physical behaviors. Nevertheless, even the Hamiltonian (9) for light rays can be constructed by simply adding the em momentum interaction term $\mathbf{v} \cdot \mathbf{Q} = (\mathbf{c}/n) \cdot (n^2 - 1)\omega\mathbf{u}/c^2 = \mathbf{k} \cdot \mathbf{u}(1 - n^{-2})$ to the standard term $(c/n)k$. Thus, beyond any specific result, it should be apparent already that the presence of the em momentum in the interaction terms of the Hamiltonians of all these effects represents a unifying aspect that is more than a mere coincidence.

The mentioned achievements have been possible only after establishing the link of $\mathbf{Q}$ with the em momentum, and thus we believe that they show a scenario where the relevance the em momentum in classical and quantum physics is reaffirmed and highlighted. In closing, in Section 6 we mention, as an example, an effect of the Fizeau-Aharonov-Bohm type and tentatively discuss a test of this effect for light rays passing by and encircling a rotating cosmic object, whose periphery is characterized by a flow $\mathbf{u}$ analogous to the vector potential $\mathbf{A}$ of the standard AB effect.

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